7. *Conclusions.*

(1) Recombination appears to take place according to the simple law

$$\frac{dn}{dt} = -\alpha n^2,$$

or more generally

$$\frac{dn_1}{dt} = -\alpha n_1 n_2 = -\frac{dn_2}{dt}.$$

(2) The coefficient of recombination varies with the pressure and throughout a considerable range of pressure is proportional to the pressure.

In conclusion I desire to express my appreciation of the kindly interest which Prof. Sir J. J. Thomson has shown throughout this work.

The Flexure of Telescope Mirror-discs Arising from their Weight, and its Influence upon Resolving Power.

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The resolving power of a reflecting telescope is proportional to its aperture, the mirror being supposed accurately a paraboloid of revolution, so that a bundle of parallel rays pass through a geometrical focus after reflection. Regarded from this point of view alone it is of advantage, in the construction of a reflecting telescope, to make the mirror of as large a size as the mechanical difficulties incident to the construction of large mirrors will allow, difficulties, that is to say, such as that of obtaining the glass of the necessary homogeneity throughout, in order to avoid distortion owing to changes of temperature, and the difficulties of grinding, shaping, and polishing. There is, however, another factor to be considered which affects the problem. The mirror, even if perfect in other respects, will, when partially supported, be distorted by its weight, to a greater or less extent according to the nature of the support. This distortion will cause a diminution in the resolving power, the effect of which will evidently increase with the size of the mirror, and so will tend to counteract the advantage accruing from the larger aperture. It is conceivable even that there will, for any given method of support, be a critical aperture, an increase of the size of the mirror beyond which will actually produce a decrease instead of an increase in the resolving power. The object of the present paper is, firstly, to calculate the nature and amount of the distortion which is produced by the weight for various methods of support, and, secondly, to investigate to what extent this distortion will affect the resolving power of the instrument, and whether any limitation is thereby placed upon the size of the mirrors which are likely to be practically attainable. The types of support which are considered are necessarily comparatively simple and somewhat ideal: the difficulties of the mathematical analysis impose these limitations, but it will be seen that they are sufficient to enable us to give a definite answer to the problem as to whether the critical size of aperture is one which is likely to be reached in the construction of large mirrors, or as to whether this critical size is so large as to be of no practical significance.

Since the equations of elastic equilibrium are linear it is sufficient to consider the flexure of the disc for the two cases in which it is horizontal and vertical. The case in which it is inclined at any angle can then be obtained by a combination of these. The notation used throughout will be that used by Prof. Love in his 'Mathematical Theory of Elasticity.'*

We will discuss first the case in which the plane of the disc is horizontal, and will treat it as a circular plate of uniform thickness. This is nearly true for large mirrors and will be a sufficient approximation for the present discussion. We consider several different supports.

(1) The disc is held in a horizontal position, being supported at its centre, its edge being free.

We obtain the solution by a combination of various stress systems, which together are to give the correct body forces and are to be so adjusted as to satisfy the terminal conditions.

The conditions in this case are that the normal displacement at the centre should vanish, and that the stress resultants, and stress couples, T, S, G,† should vanish at the edge.

We use the following systems of stress—

(α) A system in which all the stress components vanish except

$$Z_z = w(z+h),$$

w being the weight of the plate per unit volume, and the z axis being vertically upwards, and the origin in the central plane. $2h$ is the thickness of the plate.

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* Camb. Univ. Press, 1906, 2nd Edition.

† *Vide* Love, *loc. cit.*, § 294.

The corresponding displacements may be taken to be given by

$$u = -\frac{\sigma w}{\mathrm{E}}(z+h)x, \qquad v = -\frac{\sigma w}{\mathrm{E}}(z+h)y,$$

$$w = \frac{w}{2\mathrm{E}}\{z^2+2hz+\sigma(x^2+y^2)\}.$$

Also all stress resultants are zero.

The axes of x and y are in the plane of the plate; u, v, w are the components of displacement parallel to the three axes; σ, E, as usual, denote Poisson's Ratio and Young's Modulus of Elasticity for the material of the plate.

(β) A stress-system in which there is a pressure $2wh$ on the face $z = h$ of the plate. The solution for a plate bent by a uniform pressure over one face is given in Love, § 307; we must in the solution there given put $p = 2wh$ and so obtain

$$u = -\frac{1+\sigma}{\mathrm{E}} \cdot \frac{wx}{4h^2}[(2-\sigma)z^3-3h^2z-2h^3-\tfrac{3}{4}(1-\sigma)z(x^2+y^2)],$$

$$v = -\frac{1+\sigma}{\mathrm{E}} \cdot \frac{wy}{4h^2}[(2-\sigma)z^3-3h^2z-2h^3-\tfrac{3}{4}(1-\sigma)z(x^2+y^2)],$$

$$w = \frac{1+\sigma}{\mathrm{E}} \cdot \frac{w}{8h^2}[(1+\sigma)z^4-6h^2z^2-8h^3z+3(h^2-\sigma z^2)(x^2+y^2) \\ -\tfrac{3}{8}(1-\sigma)(x^2+y^2)^2].$$

$$\mathrm{T}_1 = wh^2 = \mathrm{T}_2, \quad \mathrm{S}_1 = 0, \quad \mathrm{N}_1 = whx, \quad \mathrm{N}_2 = why.$$

$$\mathrm{G}_1 = \frac{wh}{8}\{(3+\sigma)x^2+(1+3\sigma)y^2\}+\frac{3-\sigma}{10}wh^3.$$

$$\mathrm{G}_2 = \frac{wh}{8}\{(1+3\sigma)x^2+(3+\sigma)y^2\}+\frac{3-\sigma}{10}wh^3.$$

$$\mathrm{H}_1 = -\frac{1-\sigma}{4}whxy.$$

These give, in the central plane $z = 0$, a radial displacement of amount $\frac{1+\sigma}{2\mathrm{E}}whr$, and a normal displacement

$$\mathsf{W} = -\frac{wh}{32\mathrm{D}}(x^2+y^2)\left(x^2+y^2-\frac{8h^2}{1-\sigma}\right),$$

where $\mathrm{D} = \tfrac{2}{3}\mathrm{E}h^3/(1-\sigma^2)$ is the flexural rigidity of the plate.

If the radius of the plate be b, we have also at the edge

$$\mathrm{G} = \frac{3+\sigma}{8}whb^2+\frac{3-\sigma}{10}wh^3,$$

$$\mathrm{T} = wh^2, \quad \mathrm{S} = 0, \quad \mathrm{N} = whb.$$

(γ) We need other solutions which give no resultant body forces in order to adjust the boundary conditions.

We may take

$$u = -\frac{1-\sigma}{2\mathrm{E}} whx, \quad v = -\frac{1-\sigma}{2\mathrm{E}} why, \quad w = -\frac{\sigma}{\mathrm{E}} whz,$$

giving $$\mathrm{G} = 0, \quad \mathrm{N} = 0, \quad \mathrm{T} = wh^2.$$

For this system the normal displacement of the central plane vanishes, but there is a radial displacement in the plane of amount $-\frac{1-\sigma}{2\mathrm{E}} whr$.

(δ) We finally take the solution corresponding to a system of generalised plane stress, given in Love, §§ 303 and 304.

The components of displacement in the central plane vanish. If $\mathbf{W}$ is the normal displacement of the central plane it is shown that $\mathbf{W}$ satisfies

$$\nabla^2\nabla^2\mathbf{W} = 0,$$

where ∇^2 is Laplace's Operator in two dimensions, and that the stress resultants are derived, when $\mathbf{W}$ is known, from the formulæ

$$\mathrm{T} = \mathrm{S} = 0, \quad \mathrm{N} = -\mathrm{D}\frac{d}{dr}(\nabla^2\mathbf{W}).$$

$$\mathrm{G} = -\mathrm{D}\nabla^2\mathbf{W} + \frac{\mathrm{D}}{r}(1-\sigma)\frac{d}{dr}\left(\mathbf{W} + \frac{8+\sigma}{10(1-\sigma)} h^2\nabla^2\mathbf{W}\right).$$

We have $\mathbf{W}$ given by an expression of the form

$$\mathbf{W} = \frac{\mathrm{A}}{2}\left\{\frac{r^2}{2}(\log r - \tfrac{1}{2}) - \frac{r^2}{4}\right\} + \frac{\mathrm{B}r^2}{4},$$

since $\mathbf{W}$ vanishes at the centre. Therefore

$$\mathrm{G} = -\mathrm{D}\{\mathrm{A}\log r + \mathrm{B}\} + \frac{\mathrm{D}(1-\sigma)}{r}\left\{\frac{\mathrm{A}r}{2}(\log r - \tfrac{1}{2}) + \frac{\mathrm{B}r}{2} + \frac{1}{10}\cdot\frac{8+\sigma}{1-\sigma} h^2 \frac{\mathrm{A}}{r}\right\}.$$

Combining these four solutions, the conditions that G, N vanish at the edge give

$$\mathrm{A} = whb^2/\mathrm{D}, \quad \mathrm{B} = whb^2\left\{-(1+\sigma)\log b + \tfrac{1}{4}(1+3\sigma) + \frac{11h^2}{5b^2}\right\}\Big/\mathrm{D}(1+\sigma).$$

The total value of $\mathbf{W}$ at any point is

$$-\frac{wh}{32\mathrm{D}} r^2\left(r^2 - \frac{8h^2}{1-\sigma}\right) + \frac{w\sigma}{2\mathrm{E}} r^2 + \frac{\mathrm{A}r^2}{4}(\log r - 1) + \frac{\mathrm{B}r^2}{4},$$

i.e. there is a downward displacement at any point of amount

$$\mathbf{W} = \frac{3w(1-\sigma)}{64\mathrm{E}h^2} r^2\{(1+\sigma)r^2 + 2(3+\sigma)b^2\} - \frac{3w(1-\sigma^2)b^2r^2}{8\mathrm{E}h^2}\log\frac{r}{b} - \frac{24+\sigma}{20\mathrm{E}} wr^2.$$

Putting $2whb^2 = \mathrm{W}$, the weight of the plate, the downward deflection at the edge is given by

$$\mathbf{W}_{\text{edge}} = \frac{3\mathrm{W}b^2}{128\pi \mathrm{E}h^3}(1-\sigma)(7+3\sigma) - \frac{\mathrm{W}}{40\pi \mathrm{E}h}(24+\sigma).$$

The second term in this expression is small compared with the first, the relative magnitudes being of the order h^2/b^2, which is comparatively small, since the usual order of magnitude of the thickness is 1/10 of the aperture.

The correctness of the first term of this expression can be proved by the use of the approximate theory which holds for the flexure of plates. This approximate theory is very useful in obtaining solutions for more complicated cases of support, when the more exact theory would either be very tedious or else impracticable.

It is shown by Love* that for a plate slightly bent by transverse forces only the stress couples at the edge are given by

$$\mathrm{G} = -\mathrm{D}\left\{\frac{\partial^2 \mathbf{W}}{\partial \nu^2} + \sigma\left(\frac{\partial^2 \mathbf{W}}{\partial s^2} + \frac{1}{\rho}\cdot\frac{\partial \mathbf{W}}{\partial \nu}\right)\right\},$$

$$\mathrm{H} = \mathrm{D}(1-\sigma)\frac{\partial}{\partial \nu}\left(\frac{\partial \mathbf{W}}{\partial s}\right), \qquad \mathrm{N} = -\mathrm{D}\frac{\partial}{\partial \nu}(\nabla^2 \mathbf{W}),$$

where $\partial/\partial\nu$ denotes differentiation along the outward normal at the edge, and $\partial/\partial s$ along the edge, ρ is the radius of curvature.

If Z is the acting force per unit area, $\mathbf{W}$ is determined by

$$\mathrm{D}\nabla^4 \mathbf{W} = \mathrm{Z}.$$

We therefore have for the case of a circular plate under the action of its own weight,

$$\frac{1}{r}\cdot\frac{d}{dr}\left[r\frac{d}{dr}\left\{\frac{1}{r}\cdot\frac{d}{dr}\left(r\frac{d\mathbf{W}}{dr}\right)\right\}\right] = \frac{\mathrm{W}}{\pi b^2 \mathrm{D}};$$

so that

$$\mathbf{W} = \frac{\mathrm{W}r^4}{64\pi b^2 \mathrm{D}} + \frac{\mathrm{A}}{2}\left\{\frac{r^2}{2}(\log r - \tfrac{1}{2}) - \frac{r^2}{4}\right\} + \frac{\mathrm{B}r^2}{4},$$

using the fact, where necessary, in the integration that $\mathbf{W}$ vanishes at the centre of the plate: so that when $r = b$ we must have, since G, N vanish at the boundary,

$$\mathrm{A} = -\frac{\mathrm{W}}{2\pi \mathrm{D}} \quad \text{and} \quad \mathrm{B} = -\frac{\mathrm{W}}{2\pi \mathrm{D}(1+\sigma)}\left\{-(1+\sigma)\log b + \frac{1+3\sigma}{4}\right\}.$$

These values of A, B agree in magnitude with those previously obtained, provided that the term in B which involves h^2 be neglected.

We therefore obtain the deflection at the edge given by the approximate expression

$$\mathbf{W} = \frac{\mathrm{W}b^2}{64\pi \mathrm{D}}\cdot\frac{7+3\sigma}{1+\sigma}.$$

* *Loc. cit.*, § 313.

(2) We now take the case in which the plate is held horizontally and clamped at the rim.

Then we must have both **W** and $d\mathbf{W}/dr$ vanishing at points on the rim; and the radial displacement in the plane of the disc must also vanish on the rim.

If we combine the system of solutions (α)-(δ) above, the resultant radial displacement is

$$\frac{wh}{2\mathrm{E}}r\{-2\sigma+1+\sigma-(1-\sigma)\},$$

and so vanishes at all points.

We may take for the solution of $\nabla^2\nabla^2\mathbf{W} = 0$ in (δ) in this case

$$\mathbf{W} = \frac{\mathrm{A}}{2}\left[\frac{r^2}{2}(\log r-\tfrac{1}{2})-\frac{r^2}{4}\right]+\frac{\mathrm{B}r^2}{4}+\mathrm{C}\log r+\mathrm{F}.$$

Since **W** is not infinite at the centre, we must have $\mathrm{C} = 0$, and in order that G may not become infinite at the centre, we must have $\mathrm{A} = 0$.

Thus the conditions $\mathbf{W} = 0$, $d\mathbf{W}/dr = 0$ at $r = b$ require that

$$\mathrm{B} = -\frac{2w\sigma}{\mathrm{E}}+\frac{wh}{4\mathrm{D}}\left(b^2-\frac{4h^2}{1-\sigma}\right), \qquad \mathrm{F} = -\frac{whb^4}{32\mathrm{D}},$$

and the depression at any point is given by

$$\mathbf{W} = \frac{\mathrm{W}}{64\pi b^2\mathrm{D}}(b^2-r^2),$$

the central depression being $\mathrm{W}b^2/64\pi\mathrm{D}$.

In this case the exact theory agrees with the approximate theory.* The reason is that no terms depending upon h^2 are involved.

(3) If the plate is supported at the edge, we must have T, S, G, **W** vanishing there. The first two evidently do so.

We can take the same solution for **W** as in (2), and the remaining conditions give

$$\mathrm{B} = \frac{3+\sigma}{4(1+\sigma)\mathrm{D}}whb^2+\frac{3-\sigma}{5(1+\sigma)\mathrm{D}}wh^3$$

and

$$-\mathrm{F} = \frac{wb^4h}{32\mathrm{D}}\cdot\frac{5+\sigma}{1+\sigma}+\frac{wb^2h^3}{480\mathrm{D}(1-\sigma^2)}(87+79\sigma+24\sigma^2),$$

giving for the value of **W** at any point, a downward displacement,

$$\mathbf{W} = \frac{wh}{32\mathrm{D}}\{b^2-r^2\}\left\{\frac{5+\sigma}{1+\sigma}b^2-r^2\right\}$$
$$-\frac{wh^3}{60(1-\sigma^2)}\left\{(3\sigma^2+23\sigma+24)r^2-\frac{(87+79\sigma+24\sigma^2)b^2}{8}\right\}.$$

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* See solution, in Love, p. 467 (ii).

The first term agrees with that obtained by means of the approximate theory.

Combining these three results we have, for the relative depressions of centre and rim,

(1) When supported at the centre,

$$\frac{3Wb^2}{128\pi Eh^3}(1-\sigma)(7+3\sigma)-\frac{W}{40\pi Eh}(24+\sigma).$$

(2) When clamped at the rim,

$$\frac{3Wb^2}{128\pi Eh^3}(1-\sigma^2).$$

(3) When supported at the rim,

$$\frac{3Wb^2}{128\pi Eh^3}(1-\sigma)(5+\sigma)+\frac{W}{640\pi Eh}(87+79\sigma+24\sigma^2).$$

Neglecting the small second term, the ratios of these expressions depend only upon the value of σ. If we put $\sigma=\frac{1}{4}$, which is approximately the case for glass, we have the relative deflections in these three cases given by the ratios

$$93:15:63.$$

The deflection is thus much smaller in the case of the clamped rim.

(4) It is to be expected that the more complete the support at the back of the mirror the less will be the effect due to the flexure. We will accordingly consider the case in which the plate is not only supported at the centre, so that there $\mathbf{W}=0$, but is, in addition, supported at the edge.

A sufficiently close value of the flexure is obtained by using the approximate theory, and since the work is much shortened by so doing, that is the course here adopted. We have

$$\nabla^4\mathbf{W}=\frac{W}{\pi b^2 D}.$$

Thus $$\mathbf{W}=\frac{Wr^4}{64\pi b^2 D}+\frac{A}{4}r^2(\log r-1)+\frac{Br^2}{4}+C\log r+D.$$

Since $\mathbf{W}=0$ at the centre, we must put $C=D=0$; and since $\mathbf{W}=0$ at the edge $r=b$, we obtain

$$\frac{W}{16\pi D}+A(\log b-1)+B=0.$$

Also $G=-D\left(\frac{d^2\mathbf{W}}{dr^2}+\frac{\sigma}{r}\cdot\frac{d\mathbf{W}}{dr}\right)$ must vanish at the edge.

The second boundary condition is accordingly

$$\frac{\sigma+3}{8\pi D}W+A\left\{(\sigma+1)\log b+\frac{1-\sigma}{2}\right\}+B(1+\sigma)=0.$$

From these two equations we obtain

$$A = -\frac{5+\sigma}{3+\sigma} \cdot \frac{W}{8\pi D},$$

$$B = -\frac{13+3\sigma-2(5+\sigma)\log b}{3+\sigma} \cdot \frac{W}{16\pi D},$$

and the value of $\mathbf{W}$ at any point of the plate is given by

$$\mathbf{W} = \frac{3(1-\sigma^2)W}{128\pi E h^3 b^2} r^2(r^2-b^2) - \frac{3W(1-\sigma^2)(5+\sigma)}{64\pi E h^3(3+\sigma)} r^2 \log\frac{r}{b},$$

which vanishes—as it should—when $r = 0, b$.

(5) We consider finally, as an example of a still more complete support, the case in which the disc of radius b is supported at the centre and the edge, and also round the circumference of a concentric circle of radius c.

We must now assume different solutions which hold according as $r \gtreqless c$.

If $r < c$ we may assume

$$\mathbf{W}_1 = \frac{Wr^4}{64\pi b^2 D} + Ar^2(\log r - 1) + Br^2,$$

$$G_1 = -D\left\{\frac{3Wr^2}{16\pi b^2 D} + 2A(\log r + \tfrac{1}{2}) + 2B + \sigma\left(\frac{Wr^2}{16\pi b^2 D} + 2A(\log r - \tfrac{1}{2}) + 2B\right)\right\}$$

$$= -D\left\{\frac{(\sigma+3)Wr^2}{16\pi b^2 D} + 2A\left((1+\sigma)\log r + \frac{1-\sigma}{2}\right) + 2B(1+\sigma)\right\}.$$

If $r > c$ we take

$$\mathbf{W}_2 = \frac{Wr^4}{64\pi b^2 D} + A'r^2(\log r - 1) + B'r^2 + C'\log r + D',$$

$$G_2 = -D\left\{\frac{(\sigma+3)Wr^2}{16\pi b^2 D} + 2A'\left((1+\sigma)\log r + \frac{1-\sigma}{2}\right) + 2B'(1+\sigma) - \frac{C'}{r^2}(1-\sigma)\right\}$$

We have here six constants occurring. These are uniquely determined from the six boundary conditions, which are that

(α) $\mathbf{W}_1$, $\mathbf{W}_2$, should vanish when $r = c$,

(β) $\dfrac{d\mathbf{W}_1}{dr} = \dfrac{d\mathbf{W}_2}{dr}$ when $r = c$, (γ) $\mathbf{W}_2 = 0$ when $r = b$.

(δ) $G_1 = G_2$ „ $r = c$, (ϵ) $G_2 = 0$ „ $r = b$.

The condition that $\mathbf{W}_1 = 0$ when $r = 0$ has been satisfied already by our choice of $\mathbf{W}_1$.

From these equations we deduce that

$$A = A' - \frac{C'}{c^2}, \qquad B = B' + \frac{C'}{c^2}\log c,$$

$$D' = -C'(\log c - 1),$$

and also that

$$A' \Big/ \left[\{(2\sigma+10)\,b^2-(\sigma+3)\,c^2\}\,(b^2-c^2)-\{4\,(\sigma+3)\,b^2-(2+2\sigma)\,c^2\}\,c^2\log\frac{b}{c} \right]$$

$$= C' \Big/ \left[b^2c^2 \left\{ (2\sigma+10)\,b^2\log\frac{b}{c}-(3+\sigma)\,(b^2-c^2) \right\} \right]$$

$$= \frac{W}{64\pi b^2 D} \Big/ \left[\left\{ 2\,(\sigma+3)\log\frac{b}{c}+2\,(1+\sigma)\log^2\frac{b}{c} \right\} c^2-(3+\sigma)\,(b^2-c^2) \right].$$

These equations suffice to determine the deflection at any point of the disc in this case.

We have still to consider the solutions of the stress equations which are appropriate to the case in which the disc is held vertically. The body forces are now in the plane of the disc, and there are no normal components. We take the x and y axes in the plane, the origin being at the centre, and the positive direction of the x axis being vertically downwards. We will consider several simple solutions, by a combination of which the solutions appropriate to various boundary conditions may be obtained.

We take the following simple solutions of the state of plane stress:—

(α) If we assume

$$X_x = -\frac{wx}{2}, \qquad Y_y = \frac{wx}{2}, \qquad X_y = -\frac{wy}{2},$$

we have a system of stresses which give a body force parallel to the x axis of amount $+w$, the weight per unit area. The remaining solutions will, therefore, be such as to give no resultant body forces.

From these values we deduce the following particular solutions for the displacements,

$$u = -\frac{w}{4E}(1+\sigma)(x^2+y^2), \qquad v = \frac{w(1+\sigma)}{2E}xy, \qquad w = 0,$$

and the stress resultants are

$$T_1 = -whx, \qquad T_2 = whx, \qquad S_1 = -S_2 = -why,$$

which give $$T = -whr\cos\theta, \qquad S = whr\sin\theta,$$

and G, H, etc., all zero.

(β) A particular solution which gives no body forces is one in which the stress resultants and the components of displacement are given by

$$u = \frac{w}{4E}(\sigma z^2-\sigma x^2-y^2), \qquad v = \frac{w}{2E}xy, \qquad w = -\frac{\sigma w}{2E}xz,$$

and $$T_1 = 0, \qquad T_2 = whx, \qquad S_1 = 0,$$

and the rest zero. Thus

$$T = whx \sin^2\theta = whr \sin^2\theta \cos\theta.$$

$$S = \tfrac{1}{2} whx \sin 2\theta = whr \sin\theta \cos^2\theta.$$

(γ) There are also a series of particular solutions for forces in the plane of the disc which are given by

$$T_1 = -T_2 = 2h\frac{\partial^2\chi}{\partial y^2}, \qquad S_1 = -2h\frac{\partial^2\chi}{\partial x\partial y},$$

$$u = -\frac{1+\sigma}{E}\cdot\frac{\partial\chi}{\partial x}, \qquad v = -\frac{1+\sigma}{E}\cdot\frac{\partial\chi}{\partial y},$$

where χ is any plane harmonic function.

These values give

$$T = 2h\left\{\cos 2\theta\frac{\partial^2\chi}{\partial y^2} - \sin 2\theta\frac{\partial^2\chi}{\partial x\partial y}\right\},$$

$$S = -2h\left\{\sin 2\theta\frac{\partial^2\chi}{\partial y^2} + \cos 2\theta\frac{\partial^2\chi}{\partial x\partial y}\right\}.$$

If we take $\chi = w(\frac{1}{6}x^3 - \frac{1}{2}xy^2)$, a plane harmonic of the third degree, we obtain

$$T = 2hw\{-\cos 2\theta . x + \sin 2\theta . y\} = -2whr\cos 3\theta,$$

$$S = 2hw\{\sin 2\theta . x + \cos 2\theta . y\} \quad = 2whr \sin 3\theta.$$

and then the displacements are

$$u = -\frac{1+\sigma}{2E}w(x^2-y^2) = -\frac{1+\sigma}{2E}wr^2\cos 2\theta,$$

$$v = \frac{1+\sigma}{E}wxy \qquad = \frac{1+\sigma}{2E}wr^2\sin 2\theta.$$

(δ) If the plate is held at the centre there must be a force applied there which is equal in magnitude to the weight of the disc. The simplest case of a singularity arising from the application of a force at a point inside the body is given in Love, § 148.

The components of displacement are

$$u = \frac{(3-\sigma)(1+\sigma)}{2E}A\log r + \frac{(1+\sigma)^2}{2E}\cdot A\frac{y^2}{r^2}, \qquad v = -\frac{(1+\sigma)^2}{2E}\cdot A\frac{xy}{r^2},$$

and the stress components are given by

$$X_x = A\frac{x}{r^2}\left(\frac{\sigma+3}{2} - (1+\sigma)\frac{y^2}{r^2}\right),$$

$$Y_y = A\frac{x}{r^2}\left(-\frac{1-\sigma}{2} + (1+\sigma)\frac{y^2}{r^2}\right),$$

$$X_y = A\frac{y}{r^2}\left(\frac{1-\sigma}{2} + (1+\sigma)\frac{x^2}{r^2}\right).$$

These are easily shown to give

$$T = 2hA\frac{\sigma+3}{2}\cdot\frac{\cos\theta}{r}, \qquad S = -2hA\frac{1-\sigma}{2}\cdot\frac{\sin\theta}{r}.$$

(1) We combine these solutions by adding to the first, the second multiplied by a constant l, m times the third, and n times the fourth. Then the stress components are found to vanish all round the edge provided that $l = -2(1+\sigma)$, $m = (1+\sigma)/4$, $n = wb^2/2\mathrm{A}$.

The coefficient in the stress resultants in the solution (δ) is then $wb^2/2\mathrm{A} \,.\, 2h\mathrm{A} = whb^2$. The resultant of the tractions on the surface of a small cavity taken round the point of application of the applied force must be equal and opposite to that force, and therefore independent of the shape of the cavity in the limit when its dimensions are taken very small.

The resultant is easily found to be $-\pi wa^2 \,.\, 2h$ parallel to the positive direction of the x axis, and is therefore equal in magnitude and opposite in direction to the weight of the body, and so is correct.

The solution thus obtained is therefore the correct solution for the case in which the disc is supported at its centre with its edge free. The displacement components are given by

$$u = \frac{w(1+\sigma)}{8\mathrm{E}}\{-3(1-\sigma)x^2+(3+\sigma)y^2-4\sigma z^2\} + \frac{wb^2}{4\mathrm{E}}(1+\sigma)^2\frac{y^2}{r^2}$$

$$+\frac{wb^2}{4\mathrm{E}}(3-\sigma)(1+\sigma)\log r + \text{const.}$$

$$v = -\frac{1-\sigma^2}{4\mathrm{E}}wxy - \frac{(1+\sigma)^2}{4\mathrm{E}}w\frac{b^2}{r^2}xy.$$

The constant occurring in the expression for u has to be determined from the condition that u should vanish when r vanishes; but on account of the logarithmic term u becomes infinite at the centre. The reason for this is that the force cannot be applied at an exact mathematical point: it must be distributed over a small area, and the value of the constant will depend upon the mode of this distribution. The difference of the displacements in the direction of the force at any two points on the disc can, however, be uniquely determined by this formula.

(2) If the disc is clamped at the edge and held vertically the displacements there must be zero. To obtain the appropriate solution we combine the cases (α), (β), (γ) above, determining the constants introduced by means of the boundary conditions; we obtain

$$u = -\frac{w}{4\mathrm{E}}(1+\sigma)(x^2+y^2) + l\frac{w}{4\mathrm{E}}(\sigma z^2-\sigma x^2-y^2) - m\frac{(1+\sigma)}{2\mathrm{E}}w(x^2-y^2) + \text{const.}$$

$$v = \frac{w(1+\sigma)}{2\mathrm{E}}xy + l\frac{w}{2\mathrm{E}}xy + m\frac{1+\sigma}{\mathrm{E}}wxy.$$

$$w = -l\frac{\sigma w}{2\mathrm{E}}xr.$$

We choose l, m so that the coefficients of x^2, y^2 in the expression for u are equal, and so that v vanishes, *i.e.*

$$l = -\frac{2(1+\sigma)}{3-\sigma}, \qquad m = -\frac{1-\sigma}{2(3-\sigma)}.$$

The constant in u is so chosen that u is zero on the rim.

We thus obtain

$$u = \frac{(1-\sigma^2)w}{2\mathrm{E}(3-\sigma)}(b^2-x^2-y^2) - \frac{(1+\sigma)\sigma w}{2\mathrm{E}(3-\sigma)}z^2,$$

$$v = 0, \qquad w = \frac{(1+\sigma)\sigma w}{(3-\sigma)\mathrm{E}}xz,$$

and the corresponding stress components are easily shown to be

$$\mathrm{X}_x = -\frac{wx}{3-\sigma}, \qquad \mathrm{Y}_z = 0,$$

$$\mathrm{Y}_y = -\frac{\sigma wx}{3-\sigma}, \qquad \mathrm{Z}_x = 0,$$

$$\mathrm{Z}_z = 0, \qquad \mathrm{X}_y = -\frac{(1-\sigma)wy}{2(3-\sigma)},$$

which give the correct body force, and so all the conditions are satisfied.

The displacement at the centre is given by

$$u = \frac{(1-\sigma^2)wb^2}{2\mathrm{E}(3-\sigma)} = \frac{\mathrm{W}(-\sigma^2)}{4\pi(3-\sigma)\mathrm{E}h}.$$

This displacement is entirely in the plane of the plate and is small compared with the displacement in the direction normal to the plate, when it is horizontal, for in the latter case there is a term involving h^3 in the denominator.

(3) For the case in which the disc is held vertically in a groove at the rim in such a manner that the normal displacements there are zero, but tangential displacements are possible, the boundary conditions are that the normal displacement and the tangential stress resultant S must vanish at the rim.

Our previous solutions cannot be combined in any manner so as to satisfy these conditions. We can obtain another solution for the case in which there are no body forces in the following way:—

We may express the stress components in terms of a function ψ, which satisfies $\nabla^4\psi = 0$, in the form

$$\widehat{rr} = \frac{1}{r^2}\cdot\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{r}\cdot\frac{\partial\psi}{\partial r}, \qquad \widehat{\theta\theta} = \frac{\partial^2\psi}{\partial r^2}, \qquad \widehat{\theta r} = \frac{-\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial\psi}{\partial r}\right).$$

We try a solution

$$\psi = \mathrm{A}r^3\cos\theta + (\mathrm{B}r^3 + \mathrm{C}r^5)\cos 3\theta,$$

and then we have

$$\widehat{rr} = 2\mathrm{A}r\cos\theta - (6\mathrm{B}r + 4\mathrm{C}r^3)\cos 3\theta = (\mathrm{X}_x\cos^2\theta + \mathrm{Y}_y\sin^2\theta + \mathrm{X}_y\sin 2\theta),$$

$$\widehat{\theta\theta} = 6\mathrm{A}r\cos\theta + (6\mathrm{B}r + 20\mathrm{C}r^3)\cos 3\theta = (\mathrm{X}_x\sin^2\theta + \mathrm{Y}_y\cos^2\theta - \mathrm{X}_y\sin 2\theta),$$

$$\widehat{\theta r} = 2\mathrm{A}r\sin\theta + (6\mathrm{B}r + 12\mathrm{C}r^3)\sin 3\theta = [\tfrac{1}{2}(\mathrm{Y}_y - \mathrm{X}_x)\sin 2\theta + \mathrm{X}_y\cos 2\theta],$$

$$\therefore\quad 2\mathrm{A}r\cos\theta + (6\mathrm{B}r + 12\mathrm{C}r^3)\cos 3\theta = [\tfrac{1}{2}(\mathrm{Y}_y - \mathrm{X}_x)\cos 2\theta - \mathrm{X}_y\sin 2\theta],$$

whence we deduce that

$$\mathrm{X}_x = 2(\mathrm{A} - 3\mathrm{B})x - 4\mathrm{C}x(x^2 + 9y^2),$$

$$\mathrm{Y}_y = 6(\mathrm{A} + \mathrm{B})x + 4\mathrm{C}x(5x^2 - 3y^2),$$

$$\mathrm{Y}_x = -2(\mathrm{A} - 3\mathrm{B})y + 12\mathrm{C}y(x^2 + y^2),$$

whence

$$\frac{\partial u}{\partial x} = \frac{1}{\mathrm{E}}\{[2\mathrm{A} - 6\mathrm{B} - 6\sigma(\mathrm{A} + \mathrm{B})]x - 4\mathrm{C}(1 + 5\sigma)x^3 - 4\mathrm{C}(9 - 3\sigma)xy^2\},$$

$$\frac{\partial v}{\partial y} = \frac{1}{\mathrm{E}}\{[6(\mathrm{A} + \mathrm{B}) - 2\sigma(\mathrm{A} - 3\mathrm{B})]x + 4\mathrm{C}(5 + \sigma)x^3 - 4\mathrm{C}(3 - 9\sigma)xy^2\},$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{2(1 + \sigma)}{\mathrm{E}}\{2(3\mathrm{B} - \mathrm{A})y + 12\,\mathrm{C}y(x^2 + y^2)\}.$$

These three equations are satisfied by

$$u = \frac{1}{\mathrm{E}}[\{(1 - 3\sigma)\mathrm{A} - 3(1 + \sigma)\mathrm{B}\}x^2 - \{(5 + \sigma)\mathrm{A} - 3(1 + \sigma)\mathrm{B}\}y^2 - \mathrm{C}(1 + 5\sigma)x^4 - 2\mathrm{C}(9 - 3\sigma)x^2y^2 + (7 + 3\sigma)\mathrm{C}y^4 + \mathrm{D}],$$

$$v = \frac{1}{\mathrm{E}}[\{(6 - 2\sigma)\mathrm{A} + 6(1 + \sigma)\mathrm{B}\}xy + 4\mathrm{C}(5 + \sigma)x^3y - 4\mathrm{C}(1 - 3\sigma)xy^3].$$

The normal component of the displacement at any point is $(ux + vy)/r$

$$= \frac{x}{r\mathrm{E}}[\{(1 - 3\sigma)\mathrm{A} - 3(1 + \sigma)\mathrm{B}\}x^2 + \{(1 - 3\sigma)\mathrm{A} + 9(1 + \sigma)\mathrm{B}\}y^2 - \mathrm{C}(1 + 5\sigma)(x^2 + y^2)(x^2 - 3y^2) + \mathrm{D}].$$

$$\mathrm{S} = 2h\left[2\mathrm{A}y + \left(\frac{6\mathrm{B}}{r^2} + 12\mathrm{C}\right)\{3y(x^2 + y^2) - 4y^3\}\right].$$

We combine this solution with the solution (α) above, for which the normal displacement at any point is

$$\frac{w(1 + \sigma)}{4\mathrm{E}} \cdot \frac{x}{r}(y^2 - x^2).$$

Then, when $r = b$, by putting $x^2 = b^2 - y^2$, and equating the coefficients of x, x^3 to zero, we obtain

$$\mathrm{A} = -\frac{w}{4}, \quad \mathrm{B} = -\frac{w(1 + \sigma)}{4(5 + \sigma)}, \quad \mathrm{C} = \frac{w(1 + \sigma)}{8b^2(5 + \sigma)}, \quad \mathrm{D} = \frac{wb^2(3 - 5\sigma)}{8},$$

so that the components of displacement at any point of the disc are given by

$$u = \frac{w}{8\mathrm{E}(5+\sigma)b^2}\{4(3\sigma^2+10\sigma-1)b^2x^2+4(11+2\sigma-\sigma^2)b^2y^2$$
$$-(1+\sigma)(1+5\sigma)x^4-6(3-\sigma)(1+\sigma)x^2y^2+(7+3\sigma)(1+\sigma)y^4\}$$
$$+\frac{wb^2}{8\mathrm{E}}(3-5\sigma).$$

$$v = -\frac{w}{\mathrm{E}(5+\sigma)}\cdot\frac{xy}{b^2}\left\{(9+2\sigma+\sigma^2)b^2-\{(5+\sigma)x^2-(1-3\sigma)y^2\}\frac{1+\sigma}{2}\right\},$$

The central deflection is given by

$$u = \frac{wa^2}{8\mathrm{E}}(3-5\sigma) = \frac{\mathrm{W}}{16\pi\mathrm{E}h}(3-5\sigma), \qquad v = 0.$$

In the case in which the plate is inclined to the vertical there will be a superposition of the two effects. There is now a component of gravity in the plane of the plate and a component normal to it. This produces the normal sag which is, in general, the chief distortion. The component of the force in the plane of the plate produces small displacements in the plane, accompanied by very slight normal displacements, which are symmetrical about and vanish on the central plane.

We can obtain some information as to the relative magnitudes of these effects from the results which have been already obtained.

When the plate is supported horizontally at the centre it has been shown that the deflection at the edge is

$$\frac{3\mathrm{W}b^2}{128\pi\mathrm{E}h^3}(1-\sigma)(7+3\sigma)-\frac{\mathrm{W}}{40\pi\mathrm{E}h}(24+\sigma).$$

The ratio of the two terms which occur in this expression is

$$\frac{16}{15}\cdot\frac{24+\sigma}{(1-\sigma)(7+3\sigma)}\cdot\frac{h^2}{b^2}.$$

If we put $\sigma = \frac{1}{4}$, and $h = \frac{1}{10}b$, which gives ratio of aperture : thickness equal to 10, we obtain ratio of the two terms to be 1 : 22·5, so that the second term is tolerably small when compared with the first.

When the plate is supported at the rim the deflection at the centre is

$$\frac{3\mathrm{W}b^2}{128\pi\mathrm{E}h^3}(1-\sigma)(5+\sigma)+\frac{\mathrm{W}}{640\pi\mathrm{E}h}(87+79\sigma+24\sigma^2),$$

and the ratio of the second term in this expression to the first under the same conditions is 1 : 54·5.

In the case in which the plate was held vertically and clamped at the edge we have seen that the central displacement is

$$\frac{W(1-\sigma^2)}{4\pi(3-\sigma)Eh},$$

and the ratio of this to the small term $(24+\sigma)W/40\pi Eh$ occurring in the normal displacement when supported at the centre is

$$\frac{1-\sigma^2}{10(3-\sigma)(24+\sigma)} = \frac{1}{711},$$

and so is very small indeed. The displacement in the plane of the disc when this is held vertical is in this case only about 1/16,000 of the normal displacement at the edge when the disc is supported horizontally at the centre.

If the disc is vertical and held at the centre the relative displacement for points on the same radius at distances r_1, r_2 from the centre is

$$u_1-u_2 = \frac{W(1+\sigma)}{16\pi Ehb^2}\{-3(1-\sigma)(r_1^2-r_2^2)\cos^2\theta+(3+\sigma)(r_1^2-r_2^2)\sin^2\theta\}$$
$$+\frac{W}{8\pi Eh}(3-\sigma)(1+\sigma)\log\frac{r_1}{r_2}.$$

If $r_1 = b = 50$ cm. and $r_2 = 1$ cm., then $u_{50}-u_1$ has a maximum value when $\theta = \frac{1}{2}\pi$, and this maximum is approximately

$$\frac{W(1+\sigma)}{16\pi Eh}(3+\sigma)+\frac{W}{8\pi Eh}(3-\sigma)(1+\sigma)\log_e 50 = \frac{W}{16\pi Eh}\,29{\cdot}3,$$

which is rather larger than the small term $W(24+\sigma)/40\pi Eh$ which occurs in **W** in the first horizontal case considered.

When the disc was vertical and supported round the rim the central deflection was $W(3-5\sigma)/16\pi Eh$. This is about 5/4 the value when the edge is clamped.

It follows that, unless the disc when horizontal is very well supported at the back, the displacement normal to the disc which is produced by its weight is very large compared with the displacement in the plane of the disc when it is held vertically.

We have now to apply the results which have been obtained to discuss the extent to which resolving power is affected by the flexure. We suppose the undistorted mirror to be accurately paraboloidal with a focal length a, and take a system of rectangular axes such that the z axis is the axis of the mirror, and the vertex is at the origin. Then the equation of the paraboloid is

$$x^2+y^2 = 4az.$$

A bundle of rays all parallel to the axis will after reflection all pass through the point of intersection of the axis with the focal plane, *i.e.* through the point $x = 0$, $y = 0$, $z = a$.

We consider now another parallel bundle, each ray of which makes a small angle θ with the axis. The ray which passes through the point (α, β, γ) of the surface has the equation

$$\frac{x-\alpha}{\theta} = \frac{y-\beta}{0} = \frac{z-\gamma}{1}.$$

Then the reflected ray may be represented by

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}.$$

Since the direction cosines of the normal to the surface are proportional to α, β, $-2a$, and since the incident and reflected rays make equal angles with the normal, and are all in the same plane, we have the conditions

$$l\alpha+m\beta-n\,.\,2a = \alpha\theta-2a;$$

and also

$$\begin{vmatrix} l & m & n \\ \alpha & \beta & -2a \\ \theta & 0 & 1 \end{vmatrix} = 0,$$

together with

$$l^2+m^2+n^2 = 1.$$

Since θ has been assumed so small that its square and higher powers may be neglected, and since also

$$\alpha^2+\beta^2 = 4a\gamma,$$

we obtain for the values of l, m, n the expressions, correct up to terms including θ,

$$l = -\frac{\alpha}{a+\gamma}+\left\{\frac{\alpha^2}{2a\,(a+\gamma)}-1\right\}\theta, \qquad m = -\frac{\beta}{a+\gamma}+\frac{\alpha\beta}{2a\,(a+\gamma)}\,\theta$$

$$n = \frac{a-\gamma}{a+\gamma}-\frac{\alpha\theta}{a+\gamma},$$

and, to the same order of approximation, the co-ordinates of the point in which the reflected ray cuts the focal plane are given by

$$x = -\frac{\theta}{2a}\cdot\frac{a+\gamma}{a-\gamma}(2a^2+\alpha^2-2a\gamma), \qquad y = -\frac{\theta}{2a}\cdot\frac{a+\gamma}{a-\gamma}\,\alpha\beta,$$

or the distance from the axis of the point in which the reflected ray through (α, β, γ) cuts the focal plane is

$$\theta\left(\frac{a+\gamma}{a-\gamma}\right)\{(a+\gamma)^2-\beta^2\}^{\frac{1}{2}}.$$

For a given value of γ this is greatest when $\beta = 0$, *i.e.* for the point in the plane through the axis which contains the direction of the bundle. For points in this plane the greatest value is at the edge, and is $\theta\,(a+\gamma_e)^2/(a-\gamma_e)$ where γ_e is the edge value.

Since even for points on the rim γ_e will be comparatively small we may take this distance to be approximately $a\theta$.

Now two points of light, one of which is on the axis and the other at a small angular distance θ, will be resolved provided that $\theta \geqq 0{\cdot}61\,\lambda/b$, where λ is the wave-length of the light, and b is the semi-aperture of the telescope, for when this condition is satisfied the angular separation of the spots is of sufficient magnitude to prevent their first diffraction maxima overlapping.

Hence if two stars are to appear distinct, the breadth of the central image for parallel rays must not exceed $0{\cdot}61\lambda a/b$.

Suppose now that the undistorted surface has for its equation

$$\xi^2+\eta^2 = 4a\zeta.$$

Then u, v, w, the components of the displacement of the point ξ, η, ζ, are functions of ξ and η, and the co-ordinates of the displaced point are given by

$$x = \xi+u, \qquad y = \eta+v, \qquad z = \zeta+w.$$

Since u, v, w are all small we can in them put $\xi = x$, $\eta = y$, so that they become functions of x and y. The equation of the deformed surface is therefore

$$(x-u)^2+(y-v)^2 = 4a\,(z-w).$$

We consider first, for simplicity, the deformation which is produced when the disc is horizontal. In this case $u = v = 0$, the only displacement being normal to the disc and symmetrical, so that we can proceed in two dimensions.

The section of the surface by the xz plane has for its equation

$$x^2 = 4a\,(z-w).$$

The equation of an incident ray parallel to the axis is $x = \alpha$, and since the direction cosines of the normal to the surface at the point α are proportional to

$$\alpha+2a\left(\frac{\partial w}{\partial x}\right)_\alpha, \qquad -2a,$$

it follows that, the reflected ray being

$$\frac{x-\alpha}{l} = \frac{z-\gamma}{n},$$

l and n are given by the equations

$$l = -4a\left(\alpha+2a\frac{\partial w}{\partial x_\alpha}\right)\Big/\left\{4a^2+\left(\alpha+2a\frac{\partial w}{\partial x_\alpha}\right)^2\right\},$$

$$n = \left\{4a^2-\left(\alpha+2a\frac{\partial w}{\partial x_\alpha}\right)^2\right\}\Big/\left\{4a^2+\left(\alpha+2a\frac{\partial w}{\partial x_\alpha}\right)^2\right\},$$

and the reflected ray is

$$x = \alpha - \frac{4a\left(\alpha + 2a\dfrac{\partial w}{\partial x_\alpha}\right)}{4a^2 - \left(\alpha + 2a\dfrac{\partial w}{\partial x_\alpha}\right)^2}\left\{z - \frac{\alpha^2 + 4aw}{4a}\right\}.$$

If $w = 0$, *i.e.* if the mirror is undistorted, there is perfect focussing at $z = a$. We suppose then that the nearest approach to focussing when the mirror is bent is on the plane $z = a + \zeta$, where ζ is a small quantity which has to be determined.

Then the corresponding value of x is given by

$$x = \alpha - \frac{4a(\alpha + 2aw_\alpha)}{4a^2 - \alpha^2 - 4a\alpha w_\alpha}\left\{\frac{4a^2 - \alpha^2 + 4a(\zeta - w)}{4a}\right\},$$

where w_α has been used to denote $(\partial w/\partial x)_\alpha$, and where squares of w have been neglected.

On expansion, retaining only first power of w, there results

$$x = -2aw_\alpha - \frac{4a\alpha(\alpha w_\alpha + \zeta - w)}{4a^2 - \alpha^2}.$$

(1) In the case when the disc is clamped at the edge it has been shown that

$$w = k(b^2 - x^2)^2, \quad \text{where} \quad k = \mathrm{W}/64\pi b^2 \mathrm{D}.$$

By differentiation of the above value of x it is found that the condition for a stationary intercept is

$$2(\alpha w_\alpha + \zeta - w) + (4a^2 - \alpha^2)\, w_{\alpha\alpha} = 0,$$

where $w_{\alpha\alpha} = (\partial^2 w/\partial x^2)_\alpha$, and then we get

$$x = 2a(\alpha w_{\alpha\alpha} - w_\alpha).$$

The condition for a stationary value becomes in the case at present under discussion,

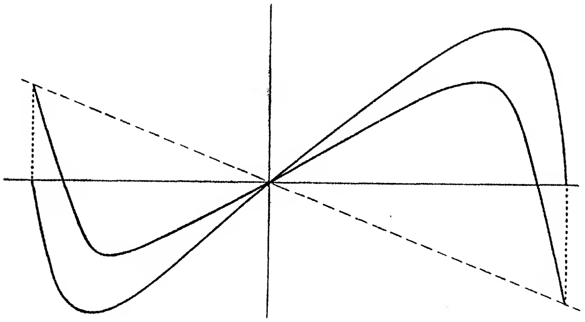
$$3\alpha^4 - 24a^2\alpha^2 + b^2(b^2 + 8a^2) - \zeta/k = 0.$$

For a given value of ζ there are two values of α. In the case in which $\zeta = kb^2(8a^2 + b^2)$, these combine to give an inflexional tangent at $\alpha = 0$.

The value of x at any point is

$$x = \frac{k\alpha}{a}\left\{(8a^2 + b^2 + 3\alpha^2)(b^2 - \alpha^2) - \frac{\zeta}{k}\right\}.$$

The part which is independent of ζ vanishes at the two edges, $\alpha = \pm b$, and the two parts add as shown in the diagram.



The position of the stationary points depends upon the value assigned to ζ, and the values at the ends may be either greater or less than the maxima. The most favourable case, giving the least breadth, will occur when the two are equal.

Retaining only the most important terms the maxima are given by

$$8a^2(b^2-3\alpha^2) = \zeta/k,$$

and to the same order are equal to $16ka\alpha^3$.

The end values when $\alpha = b$ are equal to $b\zeta/a$.

Hence the condition that this is equal to the maximum is

$$\frac{b\zeta}{16ka^2} = \left\{\frac{b^2}{3}-\frac{\zeta}{24ka^2}\right\}^{\frac{3}{2}},$$

which is satisfied by $\zeta = 2ka^2b^2$, giving a breadth of image $2b\zeta/a$, equal to $4kab^3$, and this gives the limiting condition that

$$4kab^3 < 0{\cdot}61\lambda a/b.$$

The following approximate mean values for glass are sufficient to determine the magnitude of the permissible limits for b, viz.:—

$$\rho = 3, \qquad \mathrm{E} = 6\times10^{11} \text{ C.G.S. units}, \qquad \lambda = 6\times10^{-5} \text{ cm.},$$

and

$$k = 3g\rho(1-\sigma^2)/64\mathrm{E}h^2.$$

Putting $\sigma = \frac{1}{4}$, the condition is that

$$\frac{b^4}{h^2} < \frac{3\mathrm{E}\lambda}{g\rho}\left(\frac{16}{15}\right)^2, \quad \text{or} \quad b^2 < 204{\cdot}3\,h.$$

If the mirror is so constructed that $h = \frac{1}{10}b$, then the limiting value of b is 20·4 cm., and so in this case the resolving power is lost for a very small breadth of aperture. Since the limiting condition is $b^2/h =$ const. this result is also an example of the general law that the thicker the mirror the larger is the permissible breadth of the aperture.

In the case when the disc is horizontal and supported at the edge, the value of w was found to be

$$w = \frac{\mathrm{W}}{64\pi a^2 \mathrm{D}}(b^2-r^2)\left(\frac{5+\sigma}{1+\sigma}b^2-r^2\right) = k\,(b^2-r^2)(\gamma b^2-r^2) \quad \text{(say)},$$

where k has the same value as above.

Proceeding exactly as above it is easily found that the stationary values are given by

$$4a^2\,\{(\gamma+1)\,b^2-6\,\alpha^2\} = \zeta/k:$$

and when $\alpha = b$,

$$x = \frac{kb}{a}\,\{4\,(\gamma-1)\,a^2b^2\}-\frac{b\zeta}{a}.$$

The maxima and minima are again $= 16ku\alpha^3$, and so the condition that these are equal to the end values is

$$16\,ka\alpha^3 = 16\,ka\left\{\frac{(\gamma+1)\,b^2}{6}-\frac{\zeta}{24\,ka^2}\right\}^{\frac{3}{2}} = 4k\,(\gamma-1)\,ab^3-\frac{b\zeta}{a},$$

which is satisfied by $\zeta = 2(2\gamma-1)ka^2b^2$, giving a breadth equal to $4kab^3$ exactly as before, and so—since k has the same value—the limiting value of b in order that the resolving power may still increase with increase of aperture has the same value as before.

(2) Coming now to the next case considered, in which the plate was supported both at the centre and at the edge, the value of $\mathbf{W}$ was shown to be

$$\mathbf{W} = kr^4+\frac{\mathrm{A}}{4}r^2(\log r-1)+\frac{\mathrm{B}r^2}{4},$$

where A, B were determined, and here $r = x$.

Then the intercept on the plane $z = a+\zeta$ is

$$\begin{aligned} x &= -2\,a\,\frac{w_\alpha\,(4a^2+\alpha^2)+2a\,(\zeta-w)}{4\,a^2-\alpha^2-4\,a\alpha w_\alpha} \\ &= -2a\,\alpha\left[4\,k\alpha^2+\frac{\mathrm{A}}{2}(\log\alpha-\tfrac{1}{2})+\frac{\mathrm{B}}{2}+\frac{1}{4a^2}\left(6\,k\alpha^4+\frac{\mathrm{A}}{2}\,\alpha^2\log\alpha+\frac{\mathrm{B}\alpha^2}{2}\right)+\frac{\zeta}{2\,a^2}\right], \end{aligned}$$

approximately, neglecting all terms of higher order.

The maximum value of the intercept is determined by the condition

$$12k\alpha^2+\frac{\mathrm{A}}{2}(\log\alpha+\tfrac{1}{2})+\frac{\mathrm{B}}{2}+\frac{\zeta}{2a^2}+\frac{1}{4a^2}\left\{30\,k\alpha^4+\frac{\mathrm{A}}{2}(3\,\alpha^2\log\alpha+\alpha^2)+\frac{3\,\mathrm{B}\alpha^2}{2}\right\} = 0,$$

and for this value of α,

$$x = a\left[16\,k\alpha^3+\mathrm{A}\alpha+12k\frac{\alpha^5}{a^2}+\frac{\mathrm{A}}{4\,a^2}(2\log\alpha+1)\,\alpha^3+\frac{\mathrm{B}\alpha^3}{2\,a^2}\right].$$

If we adjust so that the maximum and the value at the edge are equal and opposite, then the images from the two halves of the mirror overlap. This requires that

$$16k\alpha^3+\mathrm{A}\alpha+12k\frac{\alpha^5}{a^2}+\frac{\mathrm{A}}{4a^2}(2\log\alpha+1)\,\alpha^3+\frac{\mathrm{B}\alpha^3}{2a^2}$$

$$=2b\left[4kb^2+\frac{\mathrm{A}}{2}(\log b-\tfrac{1}{2})+\frac{1}{4a^2}\left(6kb^4+\frac{\mathrm{A}}{2}b^2\log b+\frac{\mathrm{B}b^2}{2}\right)-12k\alpha^2\right.$$

$$\left.-\frac{\mathrm{A}}{2}(\log\alpha+\tfrac{1}{2})-\frac{1}{4a^2}\left(30\,kb^4+\frac{\mathrm{A}}{2}b^2(3\log b+1)+\frac{3\mathrm{B}b^2}{2}\right)\right],$$

where the value of ζ determined by the maximum condition has been used.

Since the focal length is usually about five times the aperture, *i.e.* $a=10b$, the ratio of the terms $6kb^4/4a^2$, $4kb^2$ is $\frac{3}{8}\frac{b^2}{a^2}=\frac{3}{800}$, and so very little error will be caused by working with the approximate first terms.

Then for the determination of the value of α at the maximum we have the equation

$$16k\alpha^3+\mathrm{A}\alpha=2b\left\{4kb^2-12k\alpha^2+\frac{\mathrm{A}}{2}\left(\log\frac{b}{\alpha}-1\right)\right\}.$$

Now $$\frac{\mathrm{A}}{k}=-\frac{5+\sigma}{3+\sigma}8b^2=-\frac{168}{13}b^2=-13b^2,\text{ approx.},$$

so that $$16\alpha^3-13b^2\alpha+24\alpha^2b-21b^3=13b^3\log\alpha/b;$$

or, putting $\alpha=\lambda b$,

$$16\lambda^3+24\lambda^2-13\lambda-21=13\log\lambda.$$

By trial it is found that $\lambda=\frac{9}{10}$ nearly satisfies, so that we write

$$\lambda=0{\cdot}9+\xi,$$

where ξ is small. A second approximation gives $\xi=0{\cdot}004$, so that it is sufficiently accurate to take $\alpha=\frac{9}{10}b$ at the maximum.

Since the first terms are now comparatively small, we include also the next order terms on substitution of this value of α, and by writing $a=10b$, and

$$\frac{\mathrm{B}}{k}=-\frac{13+3\sigma-2(5+\sigma)\log b}{3+\sigma}4b^2=-\frac{55-42\log b}{13}4b^2,$$

it is found that the breadth of the image is $2\,(0{\cdot}053)\,kab^3$, and therefore the limiting condition is that

$$2\,(0{\cdot}053)\,kab^3<\tfrac{3}{5}\lambda a/b,$$

or, using the same values for the constants as before, this condition gives $b^2<1255h$, and if $h=\frac{1}{10}b$,

$$b<125{\cdot}5\text{ cm.,}\quad\text{or}\quad b<49{\cdot}4\text{ inches,}$$

and so the limiting permissible value of the radius, in this case, the thickness being $\frac{1}{10}$th of the aperture, is about 4 feet.

If we use a thicker plate we can use a mirror of larger aperture, *e.g.* if $h = \frac{1}{6}b$, then $b = 82$ inches, or the permissible radius is nearly 7 feet.

By comparison with the preceding cases, it is at once apparent how great is the advantage which is obtained simply by supporting the mirror at the centre in addition to the edge.

(3) It now remains to discuss the last mode of support considered, when the disc was horizontal, viz., when it is supported at the centre, round the circumference, and round a concentric circle of radius c. Using the values of the displacement which were found for the two portions of the disc we obtain for the values of the intercept on the plane $z = a+\zeta$ the values, according as $\alpha<c$ or $>c$ respectively,

$$x_1 = -2a\alpha\left\{4k\alpha^2+2\mathrm{A}'(\log\alpha-\tfrac{1}{2})+2\mathrm{B}'+\frac{\mathrm{C}'}{c^2}-2\frac{\mathrm{C}'}{c^2}\log\frac{\alpha}{c}\right.$$
$$\left.+\frac{1}{2a^2}\left(3k\alpha^4+\mathrm{A}'a^2\log\alpha+\mathrm{B}'\alpha^2-\mathrm{C}'\frac{\alpha^2}{c^2}\log\frac{\alpha}{c}\right)+\frac{\zeta'}{2a^2}\right\},$$

$$x_2 = -2a\alpha\left\{4k\alpha^2+2\mathrm{A}'(\log\alpha-\tfrac{1}{2})+2\mathrm{B}'+\frac{\mathrm{C}'}{\alpha^2}\right.$$
$$\left.+\frac{1}{2a^2}\left(3k\alpha^4+\mathrm{A}'\alpha^2\log\alpha+\mathrm{B}'\alpha^2-\mathrm{C}'\log\frac{\alpha}{c}\right)+\frac{\zeta'}{2a^2}\right\},$$

which are continuous at $\alpha = c$, as they should be.

We can abbreviate the work and at the same time illustrate the advantage given by the additional support if we take a special value for c. The simplest value to take is $c = b/2$, and then, by putting $\sigma = \frac{1}{4}$, we have A′, C′ given by

$$\mathrm{A}' = -4{\cdot}273\,kb^2, \qquad \mathrm{C}' = -1{\cdot}01\,kb^4,$$

and also

$$\mathrm{B}' = -\mathrm{A}'\log b-3{\cdot}196\,kb^2.$$

On investigation, it is found that the conditions cannot be satisfied by supposing the maximum deflection to fall in the inner portion, and by making it equal to the end deflection. We therefore make the maximum in the outer portion equal to the end deflection.

The condition for the maximum is

$$12k\alpha^2+\mathrm{A}'(2\log\alpha+1)+2\mathrm{B}'-\frac{\mathrm{C}'}{\alpha^2}+\frac{\zeta}{2a^2} = 0,$$

and the corresponding maximum is

$$2a\alpha\left[8k\alpha^2+2\mathrm{A}'-\frac{2\mathrm{C}'}{\alpha^2}\right],$$

and the equation to determine α is

$$2a\alpha\left(8k\alpha^2+2A'-\frac{2C'}{\alpha^2}\right) = 2ab\left[4kb^2+2A'(\log b-\tfrac{1}{2})+\frac{C'}{b^2}-12k\alpha^2\right.$$
$$\left.-A'(2\log\alpha+1)+\frac{C'}{\alpha^2}\right].$$

On substitution of the numerical values, we get

$$\frac{1{\cdot}01}{\lambda^2}+\frac{2{\cdot}02}{\lambda}+8\lambda^3+12\lambda^2-8{\cdot}546\lambda-11{\cdot}53 = 8{\cdot}546\log_e\lambda = 19{\cdot}678\log_{10}\lambda.$$

By trial it is found that $\lambda = 0{\cdot}6$ nearly satisfies, and by substituting $\lambda = 0{\cdot}6+\xi$, and retaining first powers only of ξ, a nearer approximation is $\lambda = 0{\cdot}593$. Including now the next order terms the total breadth of the image is found to be $4akb^3(0{\cdot}004)$, and the limiting condition is that $0{\cdot}16\times\frac{3g\rho}{64Eh^2}\times b^3 < \frac{3}{5}\lambda b$, so that $b^2 < 3128h$, or $b < 312{\cdot}8$ cm. < 123 inches, if $h = \frac{1}{10}b$.

The limiting value of the radius of the mirror is thus increased to about 10 feet, as compared with the 4-foot radius obtained in the case in which the support at half the radial distance is absent.

The cases which have now been discussed are sufficient to enable us to conclude that as far as the displacement normal to the plane of the disc, produced by the component of gravity in that direction, is concerned, the effect on the resolving power may be practically nullified if care is taken to support the mirror well at the back. Although the methods of support considered above are necessarily somewhat ideal, it is certain that with a mirror well supported at the back, as *e.g.* is the case in the method devised by Ritchey for the large reflecting telescope at Mount Wilson Observatory, it is possible to use mirrors of an aperture which is far beyond the limits which at present are practically attainable.

This discussion has also shown how, for a given mode of support, larger apertures may be used provided that the thickness of the mirror is increased sufficiently. The thickness to be used must of course be determined by practical considerations, such as the possibility of obtaining a homogeneous mirror, or by the question of the weight which the supports are designed to bear.

For the remainder of the discussion we shall suppose the mirror to be well enough supported at the back for the normal component of gravity to have no effect upon the resolving power, so that it only remains to discuss the effect of the displacements which occur in the plane of the disc. We will therefore consider the case in which the disc is vertical, as then these displacements will have their maximum value. The nature of the back support does not now concern us, and whatever this may be the disc must also be held in some manner at the edge and it will be impossible to prevent the displacement in the plane of the disc from taking place.

Putting $w = 0$, the equation of the mirror becomes

$$x^2+y^2-2xu-2yv = 4az.$$

The equation of the normal at (α, β, γ) is

$$\frac{x-\alpha}{p} = \frac{y-\beta}{q} = \frac{z-\gamma}{2a}\ .$$

where $\qquad -p = \alpha-u-\alpha u_\alpha-\beta v_\alpha \qquad -q = \beta-v-\beta v_\beta-\alpha u_\beta,$

and it is easily shown that corresponding to a ray incident at the point (α, β, γ) of the surface parallel to the axis, the reflected ray is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n},$$

where l, m, n are given by

$$l = \frac{4ap}{4a^2+p^2+q^2}, \qquad m = \frac{4aq}{4a^2+p^2+q^2}, \qquad n = \frac{4a^2-p^2-q^2}{4a^2+p^2+q^2};$$

and as far as the first order only in the displacements, by substituting for γ, p, q in terms of α, β, u, v, and putting $z = a+\zeta$, we have

$$x = u+\frac{(4a^2+\alpha^2-\beta^2)(\alpha u_\alpha+\beta v_\alpha)+2\alpha\beta(\beta v_\beta+\alpha u_\beta)-4a\alpha\zeta}{4a^2-\alpha^2-\beta^2};$$

and, similarly,

$$y = v+\frac{(4a^2+\beta^2-\alpha^2)(\beta v_\beta+\alpha u_\beta)+2\alpha\beta(\alpha u_\alpha+\beta v_\alpha)-4a\beta\zeta}{4a^2-\alpha^2-\beta^2}.$$

For simplicity of discussion, we will consider first the case in which $\beta = 0$, *i.e.* we consider for the present only the rays which are incident at points on the mirror in the vertical plane through its axis.

For the case of the mirror which is clamped at the edge

$$u = k(b^2-\alpha^2), \qquad v = 0,$$

where $$k = \frac{\mathrm{W}(1-\sigma^2)}{4\pi \mathrm{E}h(3-\sigma)b^2} = \frac{(1-\sigma^2)g\rho}{2\mathrm{E}(3-\sigma)},$$

and so $$y = 0, \quad \text{and} \quad x = k(b^2-\alpha^2)-\frac{4a^2+\alpha^2}{4a^2-\alpha^2}2k\alpha^2-\frac{4a\alpha\zeta}{4a^2-\alpha^2}$$

$$= k(b^2-3\alpha^2)-\frac{4\alpha\zeta}{a}.$$

Neglecting the term in α^4/a^2 and all terms of a higher order, we see that, for a given value of ζ, x is a maximum when $\alpha = -2\zeta/3ak$.

We suppose that ζ is positive: then the intercept x is positive and is equal to $kb^2+\frac{4}{3}\zeta^2/a^2k^2$.

The value of x steadily diminishes with increase of α, since ζ is supposed positive, and so the greatest negative intercept is from $\alpha = b$ and is $-2kb^2-4b\zeta/a$, so that the total breadth of the band is

$$3kb^2+\frac{4b\zeta}{a}+\frac{4\zeta^2}{3ka^2} = \frac{(2\zeta+3akb)^2}{3ka^2},$$

and to this order is least when we put $\zeta = 0$.

A similar result is obtained if ζ is supposed negative, since then the greatest negative intercept is from $\alpha = -b$.

Thus the least value of the intercept on a plane perpendicular to the axis produced by the reflection of a beam incident parallel to the axis in the vertical plane through the axis is $3kb^2$. The upper limit for b in order that the resolving power may not decrease with increase of the size of the mirror is given by

$$3kb^2 < \tfrac{3}{5}\lambda a/b \quad \text{or} \quad b^3/a < \tfrac{2}{5}\mathrm{E}\lambda(3-\sigma)/(1-\sigma^2)g\rho,$$

and if the focal length is five times the aperture, $a = 10b$, we obtain the limiting value of b to be

$$b = 379 \text{ cm.} = 149 \text{ inches},$$

and so the limiting radius of the mirror is about 12·5 feet, and quite a large size is permissible.

Whereas, in the cases previously considered in which the mirror was horizontal, the determining factor was the ratio of the aperture of the mirror to its thickness, the limiting value of the radius being found from an equation of the type b^2/h = constant; in the present case the determining factor is the ratio of the aperture to the focal length, the limiting value of b being found from an equation of the type b^3/a = constant.

If it was desired to construct a mirror of given focal length and of as large an aperture as possible, the limit for the aperture could be determined from the above. Thus, if the required focal length were 50 feet, the largest radius which may be used is 280 cm., or about 9 feet.

In the more general case when β is not zero we have for a first approximation

$$x = u+\alpha u_\alpha+\beta v_\alpha, \qquad y = v+\beta v_\beta+\alpha u_\beta.$$

Or, since $$u = k(b^2-\alpha^2-\beta^2), \quad v = 0,$$

we have $$x = k(b^2-3\alpha^2-\beta^2), \qquad y = -2k\alpha\beta.$$

Corresponding to the centre $\alpha = \beta = 0$, we have $x = kb^2$, $y = 0$, and the radius vector from this point to any other point of the patch is given by

$$k\,\{(\alpha^2+\beta^2)^2+8\alpha^2(\alpha^2+\beta^2)\}^{\frac{1}{2}},$$

and this attains its greatest value when $(\alpha^2+\beta^2)$ and α^2 have their greatest values, *i.e.* at the two points on the circumference in the vertical plane through the axis, and then $\rho = 3kb^2$. This gives the maximum extent of the swing for light incident at any point, and is the case discussed above.

In the other case considered, in which the normal component of the displacement at any point of the edge vanishes, the normal component of the displacement at any point of the disc, using the values of u, v, previously obtained, is given by

$$\frac{\alpha u+\beta v}{\rho} = \frac{w\alpha}{(5+\sigma)\,\mathrm{E}\rho}\left\{\tfrac{1}{2}(3\sigma^2+10\sigma-1)\,\alpha^2-\tfrac{1}{2}(3\sigma^2+2\sigma+7)\,\beta^2 \right.$$
$$\left. -\frac{1}{8b^2}(1+\sigma)(1+5\sigma)(\alpha^2+\beta^2)(\alpha^2-3\beta^2)\right\}$$
$$+\frac{w\alpha}{4\mathrm{E}\rho}\{(1+\sigma)(\beta^2-\alpha^2)+\tfrac{1}{2}b^2(3-5\sigma)\}.$$

The point in which the reflected ray from α, β cuts the focal plane is given by

$$x = u+\alpha u_\alpha+\beta v_\alpha, \qquad y = v+\beta v_\beta+\alpha u_\beta,$$

giving a normal component of amount

$$\frac{x\alpha+y\beta}{\rho} = \frac{u\alpha+v\beta}{\rho}+\alpha\left(\frac{\alpha}{\rho}\cdot\frac{\partial u}{\partial\alpha}+\frac{\beta}{\rho}\cdot\frac{\partial u}{\partial\beta}\right)+\beta\left(\frac{\alpha}{\rho}\cdot\frac{\partial v}{\partial\alpha}+\frac{\beta}{\rho}\cdot\frac{\partial v}{\partial\beta}\right),$$

and, since $\rho^2 = \alpha^2+\beta^2$, we obtain

$$\frac{\partial}{\partial\rho} = \frac{\alpha}{\rho}\cdot\frac{\partial}{\partial\alpha}+\frac{\beta}{\rho}\cdot\frac{\partial}{\partial\beta},$$

so that
$$\frac{x\alpha+y\beta}{\rho} = \frac{u\alpha+v\beta}{\rho}+\alpha\frac{\partial u}{\partial\rho}+\beta\frac{\partial v}{\partial\rho} = \frac{\partial}{\partial\rho}(\alpha u+\beta v),$$

and so is given by

$$\frac{w}{\mathrm{E}(5+\sigma)}\cdot\frac{\partial}{\partial\rho}\left\{\frac{3\sigma^2+10\sigma-1}{2}\alpha^3-\frac{3\sigma^2+2\sigma+7}{2}\alpha\beta^2\right.$$
$$\left.-\frac{(1+\sigma)(1+5\sigma)}{8b^2}\alpha\rho^2(\alpha^2-3\beta^2)\right\}+\frac{\partial}{\partial\rho}\left\{\frac{w(1+\sigma)}{4\mathrm{E}}\alpha(\beta^2-\alpha^2)\right.$$
$$\left.+\frac{wb^2}{8\mathrm{E}}(3-5\sigma)\,\alpha\right\}$$
$$=\frac{w}{\mathrm{E}(5+\sigma)}\cdot\frac{\alpha}{\rho}\left\{\tfrac{3}{4}(5\sigma^2+14\sigma-7)\,\alpha^2-\tfrac{3}{4}(5\sigma^2-2\sigma+9)\,\beta^2\right.$$
$$\left.-\frac{(1+\sigma)(1+5\sigma)}{8b^2}5(\alpha^2+\beta^2)(\alpha^2-3\beta^2)+\frac{b^2}{8}(3-5\sigma)(5+\sigma)\right\}.$$

Or, putting $\alpha = \dfrac{\rho}{\sqrt{(1+\lambda^2)}}$, $\beta = \dfrac{\lambda\rho}{\sqrt{(1+\lambda^2)}}$, where λ is constant for a definite radial direction, the normal component is

$$\frac{w}{\mathrm{E}(5+\sigma)} \cdot \frac{\rho^2}{\sqrt{(1+\lambda^2)}} \left\{ \tfrac{3}{4}(5\sigma^2+14\sigma-7)\frac{1}{1+\lambda^2} - \tfrac{3}{4}(5\sigma^2-2\sigma+9)\frac{\lambda^2}{1+\lambda^2} \right.$$
$$\left. -\frac{(1+\sigma)(1+5\sigma)}{8b^2} 5\rho^2 \frac{1-3\lambda^2}{1+\lambda^2} + \frac{b^2}{8\rho^2}(5+\sigma)(3-5\sigma) \right\}.$$

The variation of this expression with ρ for a definite value of λ, *i.e.* along a definite radius, is

$$-\frac{w\rho}{\mathrm{E}(5+\sigma)b_2} \cdot \frac{1}{(1+\lambda^2)^{3/2}} \tfrac{9}{32} \{17b^2+47\lambda^2b^2+25(1-3\lambda^2)\rho^2\},$$

and the radial component of the intersection with the focal plane is found to be, on substitution for σ,

$$-\frac{w}{\mathrm{E}(5+\sigma)b^2} \cdot \frac{1}{(1+\lambda^2)^{3/2}} \tfrac{9}{128} \{2(17+47\lambda^2)b^2+25(1-3\lambda^2)\rho^4-\tfrac{49}{3}b^4\}.$$

There are two cases to be considered, according as $\lambda^2 \gtrless \frac{1}{3}$.

(α) If $\lambda^2 < \frac{1}{3}$, the radial intercept continually decreases with increase of ρ, so that the extreme values arise from the centre and circumference, and are

$$\frac{w}{\mathrm{E}(5+\sigma)} \cdot \frac{1}{(1+\lambda^2)^{3/2}} \tfrac{147}{128} b^2 \text{ at the centre}$$

and

$$-\frac{w}{\mathrm{E}(5+\sigma)} \cdot \frac{1}{(1+\lambda^2)^{3/2}} (3+\tfrac{171}{128}\lambda^2) b^2 \text{ at the edge,}$$

and the maximum extent of the displacement is the difference of these, and is

$$\frac{w}{\mathrm{E}(5+\sigma)} \cdot \frac{1}{(1+\lambda^2)^{3/2}} \left(3+\frac{5\lambda^2}{2}\right) b^2,$$

very nearly.

This expression decreases with increase of λ and so has its maximum value when $\lambda = 0$, *i.e.* in the plane $\beta = 0$, and is then $3wb^2/\mathrm{E}(5+\sigma)$.

(β) If $\lambda^2 > \frac{1}{3}$, then the radial intercept has a stationary value for the value of ρ given by

$$\rho^2 = \frac{17+47\lambda^2}{25(3\lambda^2-1)} b^2,$$

and this value of ρ will be less than b provided that $\lambda^2 > \frac{3}{2}$.

Thus if $\lambda^2 < \frac{3}{2}$, the radial intercept continuously decreases from $\rho = 0$ to $\rho = b$. If, however, $\lambda^2 > \frac{3}{2}$ it reaches a minimum for a certain value of ρ which is less than b and is given by the above equation. After this it will commence to increase again, and the extreme values are at the centre and at this minimum.

For this value of ρ the intercept is

$$\frac{w}{\mathrm{E}(5+\sigma)} \cdot \frac{1}{(1+\lambda^2)^{3/2}} \cdot \frac{9b^2}{128} \left\{ -\frac{(17+47\lambda^2)^2}{25(3\lambda^2-1)} + \frac{49}{3} \right\},$$

and the maximum breadth of the image on the focal plane in the definite direction, λ, is

$$\frac{w}{E(5+\sigma)} \cdot \frac{1}{(1+\lambda^2)^{3/2}} \cdot \frac{9b^2}{128} \cdot \frac{(17+47\lambda^2)^2}{25(3\lambda^2-1)}.$$

Considering now the variation of this expression with λ^2, as λ^2 increases from $\frac{3}{2}$ to ∞, we find that it continually decreases, and so has its greatest value when $\lambda^2 = \frac{3}{2}$, viz.,

$$\frac{225\sqrt{2}}{128\sqrt{5}} \cdot \frac{wb^2}{E(5+\sigma)}.$$

This is less than the greatest value when $\lambda^2 < \frac{3}{2}$ and so the limiting condition for b is that

$$\frac{3wb^2}{E(5+\sigma)} < \frac{3}{5}\frac{\lambda a}{b}, \quad \text{or} \quad \frac{b^3}{a} < \frac{21}{20}\frac{\lambda E}{g\rho},$$

and on writing $a = 10\,b$ this gives a limiting value for b of 358 cm., or 141 inches, so that the disc may have with this mode of support a radius of nearly 12 feet.

This method of support corresponds closely to that used by Ritchey, who, in addition to the back support which prevented the normal displacements from being appreciable, placed the mirror in a circular metal band supported on the outside. In this way, there can be no normal displacements at the edge, but there is perfect freedom for tangential displacements to occur.

It has been assumed in the discussion immediately preceding that the mirror is held vertically. The displacements in this case necessarily attain their greatest possible values, and so if the radius is less than the critical value for this case, it will be so for any angle of inclination.

The results obtained in the present discussion may be briefly summarised as follows:—

1. The components of the displacement of any point of the mirror-disc arising from its weight have been calculated for various modes of support, both in the case when the mirror is held horizontally, and also when it is held vertically.

2. It is found that in all cases the displacements in the former case are very much larger than in the latter, and that if the mirror were not well supported at the back, the resolving power would begin to decrease as the aperture was increased for comparatively small breadths of aperture. The more complete is the back support, the larger of course is the critical size of mirror, and with simple forms of support this critical size is far larger than is likely to be practically reached.

3. It is also shown that for any given mode of support the critical size of the mirror may be increased by increasing at the same time its thickness.

4. When the mirror is held vertically the displacements are small, but the support cannot be so arranged as to get rid of them, as was the case with the normal component. It is shown, however, that the critical size of mirror is comparatively large, very much larger in fact than any mirrors at present in use, the limiting radius being somewhere in the neighbourhood of 12 feet in the two cases which are discussed in detail, when the telescope is so constructed that its focal length is five times its aperture and the aperture ten times the thickness of the mirror. For a given mode of support a larger aperture may be used by increasing the focal length.

On the Electrical Emissivity and Disintegration of Hot Metals.

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Introductory.

In February, 1912, the authors communicated to the Royal Society an account of some experiments on the emission of electricity from carbon at high temperatures.* In this investigation all the experiments were conducted at atmospheric pressure, and some evidence was brought forward that under these conditions the carriers of electricity appeared to consist almost wholly of "sputtered" matter, and that corpuscles—the carriers of negative electricity in high vacua—played in these experiments but a minor or at any rate an indirect part.

It was shown that, even in the absence of any applied potential, it was possible for charged particles to escape in sufficient number from a hot carbon surface to give rise, at high temperatures, to currents of ammeter rather than electrometer magnitude. The particles appeared to be emitted with considerable velocity and the emissivity of the radiating surface sending out the particles seemed to depend primarily on the temperature alone. It was, however, influenced to some degree also by the nature of the surrounding gas. Depending on this emissive property of carbon, a kind of ionic dynamo was constructed, capable of lighting intermittently a small group of glow-lamps (2 volts, 3 ampères).

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* Harker and Kaye, 'Roy. Soc. Proc.,' 1912, A, vol. 86, p. 379.